Solving the Incomplete Markets Model with Aggregate Uncertainty using Explicit Aggregation

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Abstract

We propose a method to solve models with heterogeneous agents and aggregate uncertainty. The law of motion describing aggregate behavior is obtained by explicitly aggregating the individual policy rule. The algorithm is simpler and faster than existing algorithms that rely on parameterization of the cross-sectional distribution and/or a computationally intensive simulation step. Explicit aggregation establishes a link between the individual policy rule and the set of necessary aggregate state variables, an insight that can be helpful in determining what state variables to include in other algorithms as well.

Key Words: numerical solutions, projection methods

JEL Classification: C63, D52

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1 Introduction

The behavior of individual agents in DSGE models with aggregate and idiosyncratic risk depends on perceived laws of motions of prices and/or aggregate variables that are, in equilibrium, consistent with the behavior of the individuals. The algorithm developed by Krusell and Smith (1998) finds solutions for parameterized individual policy rules and *separately* parameterized laws of motion for aggregate variables. The individual policy rules describe optimal behavior conditional on the aggregate laws of motions and the aggregate laws of motion provide a close fit for the behavior of the aggregates in a simulated panel that is generated using the individual policy rules. Algan, Allais, and den Haan (2008, 2009) and Reiter (2009) parameterize the cross-sectional distribution, which is used to calculate next period's aggregate moments by numerically integrating over the individual choices. These algorithms have in common that (i) an additional function related to an *aggregate* variable, like a moment or the distribution, is separately parameterized and (ii) information about the cross-sectional distribution—obtained by simulating a panel or by parameterizing the distribution—is used to establish a link between the individual and aggregate behavior.

The algorithm developed in this paper establishes the consistency between individual and aggregate behavior in a much more direct manner, namely by explicit aggregation of the individual policy rules. The direct link not only simplifies the calculations considerably, but it is also useful in itself, since it makes clear what information about the aggregate economy should be included in the set of state variables.

To clarify the algorithm we abstract, for the moment, from aggregate and idiosyncratic uncertainty. Consider the following simple model in which all agents are identical except for their initial capital stock. Agents face a standard intertemporal optimization problem taking the return on capital as given. The return on capital is a function of the aggregate capital stock only. We parameterize the individual policy function as

$$k' = \Psi_0(s) + \sum_{i=1}^{I} \Psi_i(s) k^i, \tag{1}$$

where s is a vector containing the aggregate state variables. From equation (1) it can

be seen that the individual policy rule is assumed to be a polynomial in k, but that the dependence of k' on s is not restricted. Using monomials as in equation (1) simplifies the exposition. In section 4, we show that other basis functions such as the elements of orthogonal polynomials or B-splines could be used as well.

A key step in models with heterogeneous agents is to establish a law of motion for aggregate capital, K. Given the expression in equation (1), this aggregate law of motion for K follows directly from aggregating the individual policy rule. That is,

$$K' = \Psi_0(s) + \sum_{i=1}^{I} \Psi_i(s) M(i) \tag{2}$$

where K' is next period's per capita capital stock and $M(i)$ is the cross-sectional average of k^i with $K = M(1)$. Note that we need an expression for the average *level* of the capital stock and not, for example, for the average of the log capital stock. Consequently, the left-hand side of equation (1) has to be equal the level of k'. In addition, explicit aggregation requires that the coefficients of the monomials, $\Psi_i(s)$, depend only on the aggregate state variables, s, and not on k.

Equation (2) makes clear that the I cross-sectional moments corresponding to the I monomials, k^i, are required as inputs for predicting K'. That is, the aggregate set of state variables, s, should include $M(1)$ through $M(I)$ and contains, thus, as many aggregate moments as there are basis functions in the approximating individual policy function. But if the first I cross-sectional moments are state variables, then we need aggregate laws of motions to predict these moments as well, since they appear as arguments in next period's policy function. If we had individual policy rules for $(k')^j$, $j = 1, \ldots, I$, then one could get the corresponding aggregate policy rules by explicit aggregation. One way to get a policy rule for $(k')^j$ for $j > 1$ is to use the one that is implied by the approximation for k' given in equation (1). This is a polynomial of order $I^j > I$, which means that additional moments would have to be added to s. Then additional policy rules would be needed to predict these additional moments, which in turn would introduce more state variables. Without modification, a solution based on explicit aggregation requires including an infinite number of moments as state variables whenever $I > 1$.

The key approximating step of our algorithm is to break this infinite regress problem and to construct separate approximations to the policy rules for $(k')^j$ by projecting $(k')^j$ on the space of the first I monomials. Thus,

$$\left(k'\right)^j = \Psi_{(k')^j,0} + \sum_{i=1}^{I} \Psi_{(k')^j,i}(s)k^j, \quad 1 < j \leq I. \tag{3}$$

The coefficients of the approximating functions in (1) and (3) can now be solved for with standard projection techniques.

The algorithm does not need a complete characterization of the cross-sectional distribution and, thus, does not have to rely on simulation procedures or a parameterization of the cross-sectional distribution to generate this information. The individual policy rules make clear what aspects of the cross-sectional distribution are needed to construct aggregate laws of motions. Those are the first I moments. By directly approximating the policy rules for $(k')^j$ with $1 \leq j \leq I$ we can get—using the equations of the model and explicit aggregation—a law of motion that describes the joint behavior of these J moments that is consistent with individual behavior. The algorithm, therefore, captures the information about the cross-sectional distribution that is needed to solve the model.

2 Model to solve

First-order and equilibrium conditions. Our numerical solution to the incomplete markets economy with aggregate uncertainty described in den Haan, Judd, and Juillard (2009) consists of individual policy functions, $k'(\varepsilon, k, a, M; \Psi)$, where ε is the (exogenous) individual employment status,[1] k the individual capital stock, a the exogenous aggregate state, Ψ the coefficients of the policy function, and M a set of cross-sectional means of k^j, $1 \leq j \leq I$, measured at the beginning of the period after the new employment status has been observed. We condition the cross-sectional moments on the employment status. As will become clear below, this is a natural thing to do for our algorithm, but it is not

[1] In equations, ε takes on the value 1 when the agent is employed and the value 0 when the agent is unemployed. As a subscript of a variable, ε is set equal to e when the agent is employed and equal to u when the agent is unemployed.

necessary. That is, instead of using, for example, the mean capital stocks of the employed and the unemployed, we could use instead just the per capita capital stock.

The standard projection procedure to solve for Ψ consists of the following three steps.

1. Construct a grid of the state variables.

2. At each grid point, define an error term, v, given values for ε, k, a, and M as

$$\begin{aligned} v(\varepsilon,k,a,M;\Psi) &= \tfrac{1}{c} - \textstyle\sum_{\varepsilon',a'}\left[\frac{\beta(r'+1-\delta)}{c'}\right]\pi_{aa'\varepsilon\varepsilon'} \\ &= \tfrac{1}{(r+1-\delta)k+wl-k'(\varepsilon,k,a,M;\Psi)} \\ &\quad - \textstyle\sum_{\varepsilon',a'}\left[\frac{\beta(r'+1-\delta)}{(r'+1-\delta)k'(\varepsilon,k,a,M;\Psi)+w'l'-k'(\varepsilon',k',a',M';\Psi)}\right]\pi_{aa'\varepsilon\varepsilon'} \\ &= \tfrac{1}{(r+1-\delta)k+wl-k'(\varepsilon,k,a,M;\Psi)} \\ &\quad - \textstyle\sum_{\varepsilon',a'}\left[\frac{\beta(r'+1-\delta)}{(r'+1-\delta)k'(\varepsilon,k,a,M;\Psi)+w'l'-k'(\varepsilon',k'(\varepsilon,k,a,M;\Psi),a',M';\Psi)}\right]\pi_{aa'\varepsilon\varepsilon'} \end{aligned} \tag{4}$$

with

$$\begin{gathered} l = (1-\tau)\bar{l}\varepsilon + \mu(1-\varepsilon),\ l' = (1-\tau')\bar{l}\varepsilon' + \mu(1-\varepsilon'), \\ r = \alpha a\left(\tfrac{K}{\bar{l}(1-u(a))}\right)^{\alpha-1},\ \ r' = \alpha a'\left(\tfrac{K'}{\bar{l}(1-u(a'))}\right)^{\alpha-1}, \\ w = (1-\alpha)a\left(\tfrac{K}{\bar{l}(1-u(a))}\right)^{\alpha},\ \ w' = (1-\alpha)a'\left(\tfrac{K'}{\bar{l}(1-u(a'))}\right)^{\alpha}, \\ \tau = \tfrac{\mu u(a)}{\bar{l}(1-u(a))}, \text{ and } \tau' = \tfrac{\mu u(a')}{\bar{l}(1-u(a'))}. \end{gathered}$$

Here K is the aggregate capital stock, u is the unemployment rate (which is determined by the aggregate exogenous state a), r is the rental rate, and w is the wage rate. If the worker is employed then he works $\bar{l}$ hours and his labor income equals $(1-\tau)w\bar{l}$. If he is unemployed then he receives μw. The first-order conditions of the agent, evaluated using the numerical solution $v(\cdot|\psi)$, correspond to the following set of conditions:

$$\begin{aligned} &v(\varepsilon,k,a,M;\Psi) \geq 0, \\ &v(\varepsilon,k,a,M;\Psi)k' = 0, \text{ and} \\ &k' \geq 0, \end{aligned} \tag{5}$$

for all possible values of ε, k, a, and M. For the standard projection procedure to work, equation (4) has to be a function of ε, k, a, and M. For this to be the case, we still need to specify the law of motion of M'.[2] Below we show how this can be

[2] Given a' and M' the values of r' and w' can be calculated since M' contains the mean capital stocks for the employed and the unemployed.

accomplished *without* relying on simulation techniques or parameterization of the cross-sectional distribution.

3. Ψ is found by minimizing some objective criterion that weighs the values of the error terms at the nodes of the grid.

Beginning and end-of-period distribution. Some agents switch employment status at the beginning of the period when the employment shock is revealed. Consequently, the end-of-period joint distribution of capital and employment status is not equal to the beginning-of-period joint distribution even though each agent still has the same amount of capital. Given the transition probabilities it is trivial to calculate (characteristics of) the beginning-of-period distribution given (characteristics of) the end-of-period distribution. For example,

$$K'_u = \frac{u(a)\pi_{aa'uu'}\widehat{K}_u + (1-u(a))\pi_{aa'eu'}\widehat{K}_e}{u(a')} \tag{6a}$$

$$K'_e = \frac{u(a)\pi_{aa'ue'}\widehat{K}_u + (1-u(a))\pi_{aa'ee'}\widehat{K}_e}{1-u(a')} \tag{6b}$$

Here K'_ε stands for next period's beginning-of-period aggregate capital stock and $\widehat{K}_\varepsilon$ for the end-of-period aggregate capital stock. Thus, solving for Ψ using the projection approach outlined above only requires that the end-of-period values of M can be calculated given the value of a and the beginning-of-period values of M.

3 Basic formulation of the algorithm

In this section, we present a basic formulation of the algorithm. We will use several assumptions that are not necessary, but these make it easier to understand the key steps of the algorithm. The first simplifying assumption is that there is no borrowing constraint. This assumption permits us to work with smooth policy functions, and will be relaxed in the next section.

Suppose that the individual policy functions for the employed and unemployed agent are parameterized as

$$k'_u = \Psi_{u,0}(s) + \sum_{i=1}^{I} \Psi_{u,i}(s) k^i \text{ and } \; k'_e = \Psi_{e,0}(s) + \sum_{i=1}^{I} \Psi_{e,i}(s) k^i, \tag{7}$$

where s is a vector containing the aggregate state variables a and M. Note that (i) the individual policy functions are polynomials in the individual state variables, but we allow for more general dependence in the employment status and the aggregate state variables and (ii) the left-hand side is the level of the capital stock and not, for example, the logarithm.[3] Below we show how to implement the algorithm if the individual policy rule is not of this form, but the logic of the algorithm is easiest understood in this particular specification.

Recall from the discussion in section 2 that we only need to be able to calculate end-of-period values of the aggregate state, given its beginning-of-period values. For the policy function given in equation (7), which is linear in the coefficients of the k^i terms, one can simply integrate to get

$$\begin{aligned} \widehat{K}_u &= \widehat{M}_u(1) = \Psi_{u,0}(s) + \textstyle\sum_{i=1}^{I} \Psi_{u,i}(s) M_u(i), \\ \widehat{K}_e &= \widehat{M}_e(1) = \Psi_{e,0}(s) + \textstyle\sum_{i=1}^{I} \Psi_{e,i}(s) M_e(i), \end{aligned} \tag{8}$$

where $M_\varepsilon(i)$ is the i^{th} uncentered moment of capital holdings with employment status ε.

The first lesson to learn from these expressions is that if the individual policy rule is an I^{th}-order polynomial, one has to include the first I moments of both types of agents as state variables. Thus,

$$M = [M_u(1), \cdots, M_u(I), M_e(1), \cdots, M_e(I)]. \tag{9}$$

The question is whether by using the aggregate laws of motions given in (8), combined with equations (7) and (4), it is possible to solve for Ψ_u and Ψ_e. The answer is in general no. With the expressions for $\widehat{K}_u$ and $\widehat{K}_e$, we can (conditional on a') calculate K' and,

[3] The discrete nature of the employment status makes it feasible to specify separate approximating functions for k' for each realization of the employment status. If individual productivity has continuous support, k' would be a polynomial in both individual state variables.

thus, next period's prices. However, next period's choices are a function of M', so we need to calculate all elements of $\widehat{M}$. This key step in our algorithm will be discussed next.

Linear policy rule. Suppose that $I = 1$, that is, the individual policy rule is linear in k. Then s is equal to $[a, K_u, K_e]$ and the expressions in (8) are—together with the value of a'—sufficient to calculate M'. Conditional on the individual policy rule being linear, the model with heterogeneous agents and aggregate uncertainty can be solved using a standard projection technique without relying on simulation procedures or complex approximations of the cross-sectional distribution.

Nonlinear policy rule. We now show that the previous result generalizes to the case for $I > 1$, *if* we make one additional approximating assumption. For simplicity, suppose that $I = 2$. From the discussion above we know that a minimum specification for s would be $s = [a, M_u(1), M_u(2), M_e(1), M_e(2)]$. This means that to determine s' we need expressions for $\widehat{M}^u(2)$ and $\widehat{M}^e(2)$. Using equation (7) with $I = 2$ we get

$$\begin{aligned}\left(k'_\varepsilon\right)^2 = \quad & \left(\Psi_{\varepsilon,0}(s)\right)^2 + 2\Psi_{\varepsilon,0}(s)\Psi_{\varepsilon,1}(s)k + (2\Psi_{\varepsilon,0}(s)\Psi_{\varepsilon,2} \\ & + \left(\Psi_{\varepsilon,1}(s)\right)^2)k^2 + 2\Psi_{\varepsilon,1}(s)\Psi_{\varepsilon,2}(s)k^3 + \left(\Psi_{\varepsilon,2}(s)\right)^2 k^4.\end{aligned} \tag{10}$$

Aggregation of this expression implies that we have to include the first four moments instead of the first two as state variables, that is,

$$s = [a, M_u(1), \cdots, M_u(4), M_e(1), \cdots, M_e(4)].$$

This means that to determine s' we need expressions for $\widehat{M}_\varepsilon(3)$ and $\widehat{M}_\varepsilon(4)$, which in turn implies that we need additional elements in s. The lesson learned is that whenever $I > 1$ one has to include an infinite set of moments as state variables to get an exact solution.

The key step in our algorithm is to break this infinite regress problem by approximating these policy rules that are needed to determine next period's aggregate state using lower-order polynomials. If we break the chain immediately at $I = 2$, then $\left(k'_\varepsilon\right)^2$ is obtained from the approximation

$$\left(k'_\varepsilon\right)^2 \approx \Psi_{\varepsilon,(k')^2,0}(s) + \Psi_{\varepsilon,(k')^2,1}(s)k + \Psi_{\varepsilon,(k')^2,2}(s)k^2 \tag{11}$$

and not from equation (10). Note that $\Psi_{\varepsilon,(k')^2,j}(s)$ is not equal to $\Psi_{\varepsilon,j}(s)$: with the $(k')^2$ subscript we indicate that the coefficients in this approximating relationship are not obtained from the $\Psi_{\varepsilon,j}(s)$ coefficients as in equation (10), but from a separate projection of $(k'_\varepsilon)^2$ on the space of included terms. The coefficients $\Psi_{\varepsilon,(k')^2,j}(s)$ are chosen to get the best fit for $(k'_\varepsilon)^2$ according to some measure. Given that the excluded terms, i.e., k^3 and k^4, are correlated with the included terms, these coefficients will also capture some of the explanatory power of the higher-order excluded terms. The key implication of using equation (11) instead of equation (10) is that aggregation of equation (11) does not lead to an increase in the set of aggregate state variables.

For $I = 2$ the numerical algorithm consists of the following steps. The variables on the grid are $[\varepsilon, k, a, M_u(1), M_u(2), M_e(1), M_e(2)]$. With the use of equations (8) and (11), the error terms defined in equation (4) can be calculated given values for $\Psi_\varepsilon(s)$ and $\Psi_{\varepsilon,(k')^2}(s)$. The algorithm chooses those values for the coefficients that minimize some objective function of the errors defined in equation (5).

To get expressions for next period's aggregate variables using explicit aggregation one has to break the infinite regress at some point. One could break it at $I = 2$ as in the example above, but one also could break it at some higher level. For example, suppose again that the individual policy rule is approximated well with a second-order polynomial. One possibility would be to set $I = 4$ and approximate k'_ε, $(k'_\varepsilon)^2$, $(k'_\varepsilon)^3$ and $(k'_\varepsilon)^4$ using fourth-order polynomials. But an alternative would be to approximate k'_ε with a second-order polynomial as above, use equation (10), i.e., the exact expression given the policy rule for k'_ε, to describe $(k'_\varepsilon)^2$, and construct approximations for $(k'_\varepsilon)^3$ and $(k'_\varepsilon)^4$ using fourth-order polynomials.

4 General formulation of the algorithm

The keystone of the analysis in the last section is the individual policy rule for the *level* of the capital stock given in equation (7). In this section, we give a more general formulation of the algorithm and relax two properties that we used in the previous section. In particular, we no longer require (i) that the approximating function is constructed using

continuous differentiable basis functions and (ii) that the left-hand side of the individual policy function is equal to the *level* of k.

General basis functions. We start by relaxing the first assumption. The individual policy functions for k'_u and k'_e are given by

$$k'_\varepsilon = \Psi_{\varepsilon,0}(s) + \sum_{i=1}^{I} \Psi_{\varepsilon,i}(s) B_i(k) \ , \quad \varepsilon \in \{u, e\}, \tag{12}$$

where $B_i(k)$ is the i^{th}-order basis function. There are many different choices for the basis functions. For example, they could be monomials as in the previous section. Our procedure allows for non-differentiable basis functions, as would be the case with B-splines.[4] Note that the basis functions in equation (12) are functions of the endogenous individual state variables only. An advantage of the algorithm we propose is the link between the individual policy functions and the included aggregate state variables, that is, each included basis function necessitates inclusion of its corresponding cross-sectional average as a state variable. For each basis function we need an auxiliary approximating relationship. That is, the complete set of approximating relationships would consist of equation (12) and

$$B_j(k'_\varepsilon) = \Psi_{\varepsilon,B_j(k'),0}(s) + \sum_{i=1}^{I} \Psi_{\varepsilon,B_j(k'),i}(s) B_i(k), \text{ for } j \in \{1, \cdots, I\} \text{ and } \varepsilon \in \{u, e\} \tag{13}$$

In principle there is, thus, nothing that prevents the algorithm from being implemented when splines are used to approximate the individual policy function, which is useful for problems with occasionally binding constraints. In practice, using splines may not work well because I is typically high when splines are used, which implies that the number of included state variables and auxiliary approximating functions are high too. The latter is not that problematic, but a large set of state variables can make it very time consuming to solve the model. A solution to this dilemma is discussed next.

[4] B-splines are simply a different way to express splines. For example, the basis functions of the linear B-spline, that implements linear interpolation, are defined as $(k - \kappa_j)/(\kappa_{j+1} - \kappa_j)$ if $\kappa_j \leq k \leq \kappa_{j+1}$, as $(\kappa_{j+2} - k)/(\kappa_{j+2} - \kappa_{j+1})$ if $\kappa_{j+1} \leq k \leq \kappa_{j+2}$, and 0 otherwise.

Using the level of k' as the LHS variable. The individual policy rules for k'_ε, like those in equation (7) or (12), fulfill two roles. They are used to model the individuals' choices and they are used—after explicit aggregation—to forecast next period's moments.[5] The analysis above assumes that the same approximation is used to fulfill both roles. Using the same policy function is attractive from a consistency point of view, but is not necessary from a numerical point of view. It may very well be the case that an accurate description of individual behavior requires a highly nonlinear functional form, but that a simpler functional can be used for aggregation, for example, because not too many agents face individual state variables in the nonlinear area of the policy function. We will refer to the policy rule that determines the choice of k' in the individual problem as the *individual* policy rule, and to the policy rule for k' used in the aggregation as the *primary auxiliary* policy rule. The restriction that the *level* of k' must be the left-hand side variable only holds for the primary auxiliary policy rule. That is, one can use *any* individual policy rule as long as one also has an additional policy rule for the level of k'.

In our implementation of the algorithm, we use piecewise-linear splines to approximate the individual policy rule,[6] but use a linear approximation for the primary auxiliary policy rule. In particular, we use a first-order Taylor expansion of the individual policy rule to construct the primary auxiliary policy rule. With piecewise-linear splines, aggregating the first-order Taylor expansion boils down to simply evaluating the individual policy functions of agents with employment status ε at the corresponding aggregate moments. That is,

$$\widehat{K}_\varepsilon = \Psi_{\varepsilon,0}(s) + \sum_{i=1}^{I} \Psi_{\varepsilon,i}(s) B_i(K_\varepsilon) \text{ for } \varepsilon \in \{u, e\} \tag{14}$$

with $s = [a, K_u, K_e]$. That is, we simply evaluate the individual policy rule for agents with employment status ε at the aggregate capital stock of agents with this employment status. Since the primary auxiliary individual policy rule is linear, we do not need any

[5] The policy rules for $(k'_\varepsilon)^j$, with $j > 1$, or $B_j(k'_\varepsilon)$, with $j \geq 1$, are *only* used to forecast next period's moments.

[6] For each set of values of ε and a, our policy rule is a three-dimensional linear spline in K_u, K_e, and k, which means that as a function of k it is a linear spline with coefficients that depend on K_u and K_e and, of course, ε and a.

further auxiliary policy rules and using just the mean capital stocks K_u and K_e as state variables is sufficient.

The reader may be concerned that using a linear primary auxiliary rule cannot lead to an accurate solution when the underlying law of motion is non-differentiable, which is the case here for the policy rule of the unemployed. As discussed in section 6.1, we can get a solution for this model that is very accurate in almost all dimensions. To improve accuracy one could add more basis functions, which for our methodology means more state variables. As an alternative we discuss a simple bias correction in section 6.1 that can improve the aggregate policy rule given in (14), but does not increase the set of state variables.

Perturbation procedures and the infinite regress problem. Preston and Roca (2007) use perturbation procedures to solve models with heterogeneous agents and aggregate risk using standard perturbation techniques. In particular, they perturb the model around the point with no idiosyncratic and no aggregate uncertainty.[7] This is a very innovative approach and quite different from the alternatives used in the literature and is also different from ours. There is one aspect, however, that the perturbation approach has in common with our algorithm. In the procedure of Preston and Roca (2007) and in ours there is a link between the individual policy rule and the included aggregate state variables. In particular, a linear solution requires the use of first-order moments and a second-order one requires the use of second-order moments. They do not discuss the infinite regress problem, but they suffer from it in the same way we do. When using a second-order perturbation, the individual choice for k' depends on second-order terms. Aggregation of this individual policy rule to obtain a law of motion for K' then implies

[7] Obviously, the model has to be modified because perturbation techniques cannot be used in the presence of occasionally binding constraints. Preston and Roca (2007) use penalty functions to capture the impact of borrowing constraints. It is an open question how accurate perturbation techniques are to solve these models. Perturbation techniques have the advantage that they are computationally less costly in the presence of many state variables. The policy functions obtained with perturbation techniques are always polynomials in the state variables. The individual policy functions used for aggregation in our approach only have to be polynomials in the individual state variables.

that second-order moments are state variables. Next-period's values of these second-order moments are integrals of $(k')^2$. Preston and Roca (2007) use a second-order approximation for this second-order term. But a second-order perturbation of a function squared is not as accurate as the square of a second-order approximation of the function itself. That is, like us, they use approximations that are less accurate for policy functions used to predict next period's higher-order moments.

Which moments to include as state variables? For our algorithm this question is equivalent to the question which primary auxiliary policy rule to use, because the shape of the primary auxiliary rules directly implies the set of moments one has to include. We consider this to be a salient feature of our procedure and an insight likely to be helpful in determining what moments to include in other algorithms too, even though these algorithms do not ascertain any link between the individual policy function used and the aggregate moments included.

In the particular solution submitted to the comparison project, we use separate linear primary auxiliary policy rules for the employed and the unemployed agent. Consequently, K_u and K_e are used as state variables. An alternative would be to have one primary policy rule that is linear in the individual capital stock and the employment status. In this case only the aggregate capital stock, K, would be a state variable as is the case in Krusell and Smith (1998). If a second-order capital term is added to this auxiliary policy rule, then there would be—as in our specification— two aggregate moments included in the set of state variables. What is better depends on whether the marginal propensity to save varies more with the wealth level or with the employment status. In this particular model, there is very little variation in the marginal propensity to save in either direction. But this discussion illustrates the usefulness of the link between the properties of the individual policy function and the included set of aggregate state variables that is brought to light by our algorithm.[8]

[8] For instance, in overlapping-generations models, the marginal propensity to save could very well vary more with age than with wealth. If the researcher faces the choice between including either lower-order moments that condition on age or additional higher-order moments that do not, then the logic of our

5 Solving the model without aggregate uncertainty

In this section, we discuss the algorithm when it is used to solve the model without aggregate uncertainty. There are two reasons to do this. First, it illustrates the simplicity of our algorithm and brings to light the advantages relative to alternative algorithms. Second, we want to make clear why our algorithm can solve models with heterogeneous agents without generating a complete cross-sectional distribution by simulating a panel as is done by Krusell and Smith (1998) and without parameterizing the distribution as is done by Algan, Allais, and den Haan (2008, 2009) and Reiter (2009).

If there is no aggregate uncertainty, then the problem boils down to finding a (constant) value for aggregate capital, K, such that the individual policy rules corresponding to the implied prices, $r = \alpha K^{\alpha-1}$ and $w = (1-\alpha)K^{\alpha}$, generate an ergodic distribution for k_ε with a cross-sectional mean equal to K.[9] The standard procedure to solve this problem consists of the following steps. First, given a value for K one solves for the individual policy functions at the implied prices. Second, given these policy rules one calculates the ergodic distribution and its cross-sectional mean. A non-linear equation solver can be used to find the fixed point.

With our algorithm, we can solve for the equilibrium value of K directly using standard projection techniques. The idea is as follows. We use I^{th}-order polynomials to approximate k'_u and k'_e. In the model without aggregate uncertainty, there are no aggregate state variables, which means that the value of I does not affect the dimension of the set of state variables and can, thus, be chosen to take on a high value. The algorithm consists of the following steps.

1. Start with guesses for the values of $K_u = M_u(1)$ and $K_e = M_e(1)$ *and* with guesses for the values of $M_u(j)$ and $M_e(j)$ for $j \in \{2, \cdots, I\}$.

2. Using the guesses for K_u and K_e solve for the coefficients of the individual policy rules, including the auxiliary policy rules: $\Psi_{\varepsilon,j}$ for $\varepsilon \in \{u, e\}$ and $j \in \{1, \cdots I\}$.

algorithm imiplies that the first option makes more sense.

[9] We assume that the aggregate labor supply is equal to 1.

This step is as easy as in the standard problem, except that one also has to obtain approximating functions for $(k'_\varepsilon)^j$. With most procedures one could vectorize this step, so that computationally it is not much more expensive to solve for the additional policy functions.

3. Standard procedures would obtain new values for $M_u(j)$ and $M_e(j)$ for $j \in \{1, \cdots J\}$ by simulating or by calculating the fixed point of the dynamic system with the cross-sectional distribution described by a histogram. With our algorithm, new values follow directly from the individual policy functions using as inputs the previous guesses for all the cross-sectional averages.

4. Iterate until the values have converged.

With these steps, we describe the algorithm as an iterative procedure. In most cases it is more efficient to think of this as a nonlinear system in the values of $M_\varepsilon(j)$ and to use an equation solver.[10]

The algorithm never simulates and does not parameterize the cross-sectional distribution. Consequently, the algorithm is much faster than existing algorithms and is potentially very useful for estimation. Although the algorithm never calculates a cross-sectional distribution, it does calculate unconditional cross-sectional moments, but only those that are aggregate state variables. These unconditional cross-sectional moments are exactly that part of the distribution that is needed to calculate next period's values of K_u and K_e by explicitly aggregating the individual policy functions.

6 Implementation and improvements

In this section, we describe a bias correction procedure that is useful in improving the algorithm in the sense of getting an accurate solution at a lower computational cost. We also provide more details on how to implement the algorithm.

[10] If one solves this system by iterating one may have to use dampening, that is, take a weighted average between the new and the old values of $M_\varepsilon(j)$ at each iteration.

6.1 State space reduction and bias correction

An advantage of the algorithm is that there is a tight link between the individual policy rules and the aggregate laws of motions. Although this link is desirable from a consistency point of view, it makes the algorithm expensive when the individual policy function is complex. Suppose that an accurate description of the individual policy rule requires an approximation with many basis functions, but that this complexity is not important for the aggregate law of motion. As pointed out above, one could then use two individual policy rules. The first is used to describe individual behavior and the second, a simpler auxiliary policy rule, is used as the basis for predicting the aggregates. If the auxiliary policy rule is a simpler policy rule, then it is at least to some extent misspecified. For example, the linear auxiliary policy rule used above misses the convex behavior induced by the borrowing constraint. This misspecification of the auxiliary individual policy rule causes a bias in the aggregate law of motion.

In terms of forecasting next period's aggregate capital stock, this bias turns out to be very small, but the bias is systematic and errors accumulate over time and result in more noticeable errors. When we assess the one-period ahead predictability of the aggregate law of motion that is implied by the linear primary auxiliary rule, then we find that the R^2 is equal to 0.999973 and 0.9999970 for K_u and K_e, respectively. Thus, the values of the R^2 suggest that our solution is extremely accurate. As pointed out in den Haan (2009), however, the R^2 is a very weak accuracy test. A much better accuracy test is to compare a long series of the aggregates from the simulated panel of individual observations with the corresponding series generated *separately* by the aggregate law of motion. This procedure would detect accumulation of small systematic mistakes. The results of the R^2 are indeed misleading; with this more powerful accuracy test we find that the maximum (average) errors are equal to 1.56% (0.98%) and 1.33% (0.90%) for the aggregate capital stock of the unemployed and the employed, respectively. It is always difficult to determine when numerical errors are too high, but relative to typical standards, these errors are quite large.[11] Moreover, as shown below, they can be improved upon substantially.

[11] See den Haan (2009) for a further discussion.

To reduce this error we could make the auxiliary policy rule more complex, but this would enlarge the state space. Instead we recommend the following simple procedure. The idea is to calculate the bias in the model without aggregate uncertainty and then use this value to adjust the aggregate law of motion in the model with aggregate uncertainty. Solving the model without aggregate uncertainty accurately can be done quickly, so this does not add substantially to the computational burden. Let $\tilde{K}_\varepsilon$ and $\widehat{\tilde{K}}_\varepsilon$ stand for the beginning and end-of-period values for the (constant) aggregate capital stocks of workers with employment status ε that are obtained from a very accurate solution of the model without aggregate uncertainty.[12,13] The aggregate law of motion implied by the auxiliary policy rule is given by

$$\widehat{K}_\varepsilon = \Psi_{\varepsilon,0}(M) + \Psi_{\varepsilon,1}(M)K_\varepsilon. \tag{15}$$

The bias correction term, ζ_ε, for the aggregate law of motion of K_ε is equal to the difference between $\widehat{\tilde{K}}_\varepsilon$ and the values implied by the approximating aggregate law of motion. That is,

$$\zeta_\varepsilon = \widehat{\tilde{K}}_\varepsilon - \Psi_{\varepsilon,0}(\tilde{M}) - \Psi_{\varepsilon,1}(\tilde{M})\tilde{K}_\varepsilon. \tag{16}$$

Adding this error correction to the solution of the model with aggregate uncertainty reduces the maximum (average) errors from 1.56% (0.98%) and 1.33% (0.90%) to 0.44% (0.12%) and 0.34% (0.12%), a quite substantial improvement.

In figures 1 and 2, we compare the realizations of K_u and K_e according to the aggregate law of motion (solid lines) with those values that are implied by the individual policy rules in a simulated panel (dotted lines). The two different aggregates are based on the same time series of realizations for a, but are otherwise generated independently. Figure 1 plots the series when no bias correction is used and figure 2 when it is used. We chose that part of the sample where the largest differences between the two series occur.

[12] $\tilde{K}_\varepsilon$ can be obtained from $\widehat{\tilde{K}}_\varepsilon$ using only the exogenously given transition probabilities using equations like the ones given in (6a) and (6b).

[13] To solve the model without aggregate uncertainty, we follow Young (2009) and construct a law of motion for the cross-sectional distribution using a very fine histogram. This law of motion can be represented as a linear system in the probabilities on the nodes. The ergodic distribution is then the normalized eigenvector corresponding to the unit eigenvalue.

From the figures we can make the following observations. First, without the bias correction the aggregate law of motion follows the movements of the simulated aggregates (that are implied by the individual policy rules) quite closely, but there is a systematic bias that widens during severe recessions. This despite having aggregate laws of motion with R^2s in excess of 0.99997. Second, there is virtually no systematic bias when the bias correction is applied.

The numerical solution generated with the procedure that uses the bias correction is much better in terms of passing the accuracy test proposed in den Haan (2009). However, in virtually all other properties of the model except the mean it is extremely similar to the solution obtained without the bias correction. That is, although the R^2 is a weak test, the test proposed by den Haan (2009) seems very demanding in that algorithms that generate solutions that are accurate in almost all aspects can still do poorly in terms of this accuracy test.

6.2 Details of the implementation

We use the procedure of endogenous grid point proposed in Carroll (2006) and, thus, specify nodes for k'. We use 250 nodes in the interval $[0, 250]$. To get more nodes close to the constraint we use equidistant nodes for $\ln(1 + k')$. For K_u and K_e we use 12 linearly spaced nodes in the intervals $[33, 42.5]$ and $[35, 43.5]$, respectively. For each set of values for ε and a, the policy function is then the three-dimensional interpolation of the policy choices at the nodes of the (k, K_u, K_e) grid. The coefficients of the policy rules are solved for by time iteration.

7 Other models

If the rental rate of capital and the wage rate are given by the standard expressions given in equation (4), then the aggregate demand for capital and labor by firms is exactly equal to the corresponding supply of each input. That is, even if numerical approximation errors lead to a errors in the aggregate capital stock, market equilibrium still holds at any point

in the state space and, thus, also during a simulation.[14]

In several other models, it may not be that straightforward to impose market equilibrium. The most common example in which this is the case is a bond economy. In this section, we first show how our algorithm can be used to solve a bond economy and second we give a particular implementation so that one can always ensure that markets clear along the simulated time path. Consider a simple endowment economy in which agents can smooth consumption by trading in one and two-period risk-free bonds. To simplify the discussion, we assume that there are no borrowing constraints but that instead there is a penalty function that limits agent's short position. The Euler equations for this problem are given by

$$\frac{q_t^1}{c_{i,t}} = \beta \mathrm{E}_t \frac{1}{c_{i,t+1}} + p(b_{i,t+1}^1) \tag{17}$$

$$\frac{q_t^2}{c_{i,t}} = \beta \mathrm{E}_t \frac{q_{t+1}^1}{c_{i,t+1}} + p(b_{i,t+1}^2) \tag{18}$$

and the budget constraint by

$$c_{i,t} + q_t^1 b_{i,t+1}^1 + q_t^2 b_{i,t+1}^2 = y_{i,t} + b_t^1 + q_t^1 b_t^2, \tag{19}$$

where q_t^j is the price of a zero-coupon j-period bond that pays one unit, $y_{i,t}$ is the endowment of agent i, $b_{i,t}^j$ are the bond holdings of agent i, and $p(\cdot)$ a penalty term.[15] Let $s_{i,t}$ be the set of individual state variables and M_t the set of aggregate state variables.

Some authors propose to include the bond prices in M_t to ensure that one can always clear markets. We prefer to only include predetermined variables in the set of state variables and instead choose to parameterize $b_{i,t+1}^j + q_t^j$ (for $j = 1, 2$). Thus,

$$b_{i,t+1}^1 + q_t^1 = d^1(s_{i,t}, m_t) \tag{20}$$

[14] It is important that market equilibrium holds at each point in a simulation. The reason is that numerical errors are unlikely to be zero on average. This means that for a long enough sample errors will accumulate to large numbers and then it is not clear how to interpret the simulated data.

[15] For simplicity the penalty of having low bond levels enters directly the utility function so that the derivative of the penalty function only enters the Euler equation.

and

$$b^2_{i,t+1} + q^2_t = d^2(s_{i,t}, m_t) \tag{21}$$

That is, instead of specifying an approximation for $b^1_{i,t+1}$ and $b^2_{i,t+1}$ we specify an approximation for a particular combination of $b^j_{i,t+1}$ and q^j_t. If $d^j(\cdot)$ satisfies the rules that are required for an auxiliary policy rule, then we can explicitly aggregate equations (20) and (21) and get values of the two bond prices that ensure market equilibrium at any point in the state space (or along a simulation). That is,

$$\int d^j(s_{i,t}, m_t)di = 0 + q^j_t. \tag{22}$$

Individual bond holdings can then be solved from (20) and (21) and consumption from the budget constraint.

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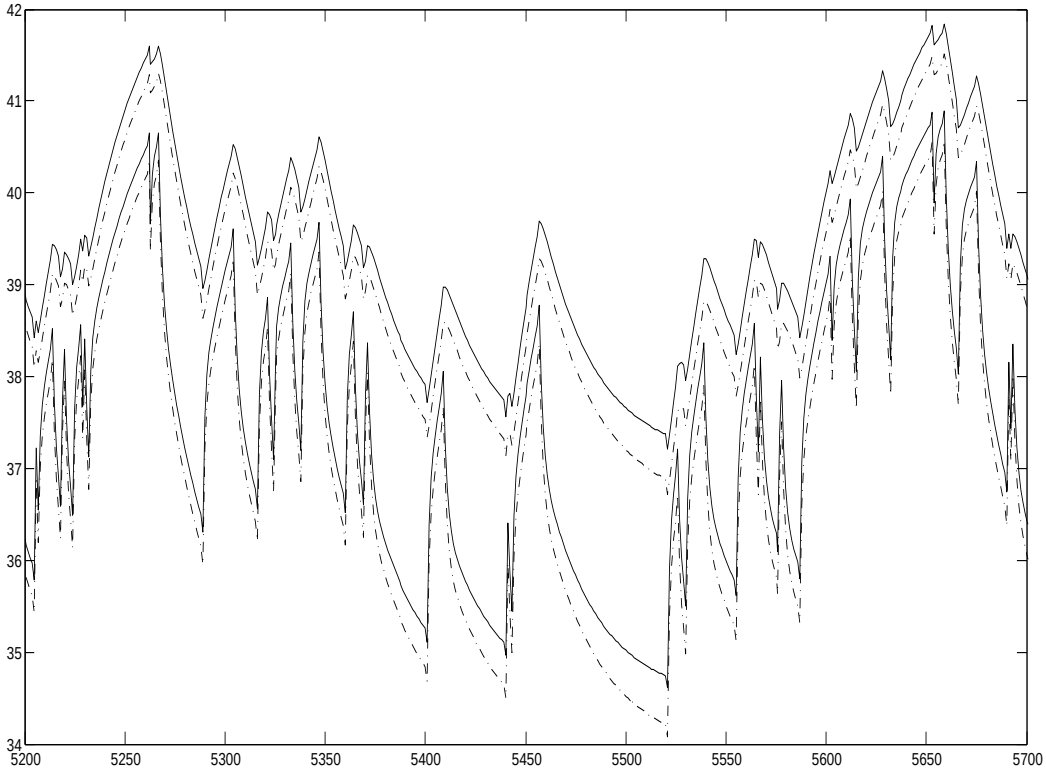
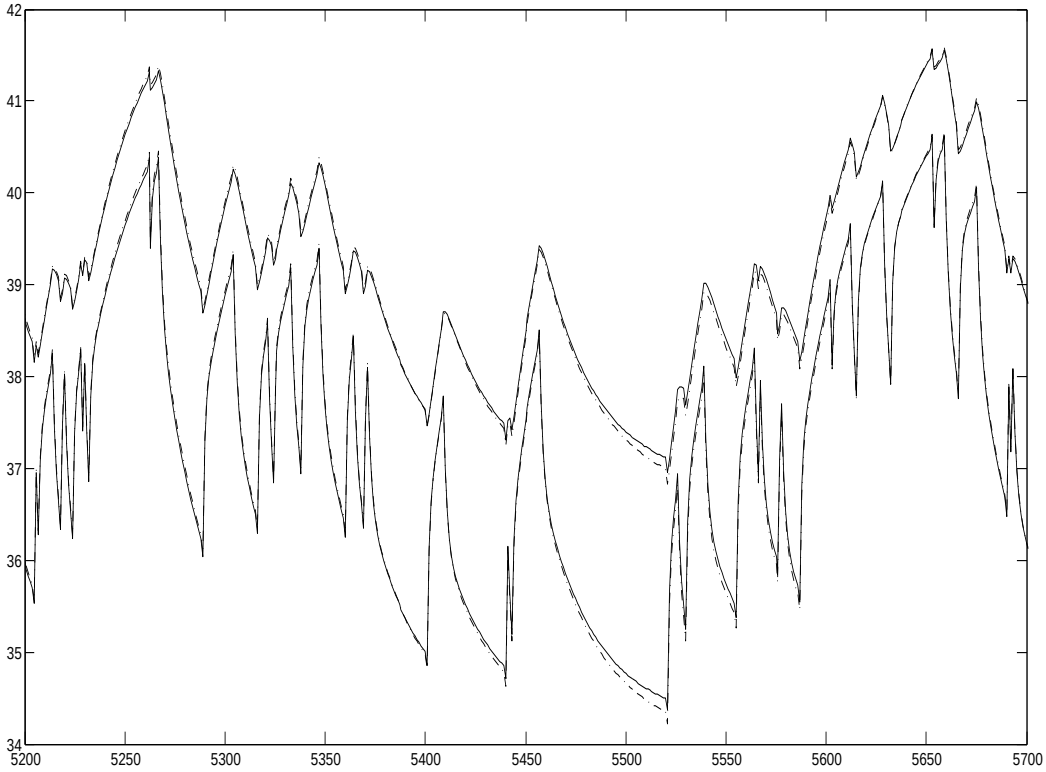
Figure 1: Simulated values of K_u and K_e without bias correction



Notes: This figure plots the aggregate capital stocks of the employed and the unemployed from the simulated panel and the corresponding series generated by the aggregate law of motion when no bias correction is implemented.

Figure 2: Simulated values of K_u and K_e with bias correction



Notes: This figure plots the aggregate capital stocks of the employed and the unemployed from the simulated panel and the corresponding series generated by the aggregate law of motion when the bias correction is implemented.