Filtering Data using Frequency Domain Filters

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Overview

- Intro
  - lag operator
  - Why frequency domain?
- Fourier transform
- Data as cosine waves
- Spectrum
- Filters & I(1) processes
- Band-pass filters
- HP filter

The big picture

In these slides we do three things

1. Make clear that *stochastic* time series can be represented as a sum of *deterministic* cosine waves

2. Learn some tools from frequency domain analysis

3. These tools will make it possible to extract from *any* time series that part associated with the frequencies we are interested (for example business cycle analysis)

Lag operator

$$
\begin{aligned}
x_{t-1} &= Lx_t \\
x_{t-2} &= Lx_{t-1} = LLx_t = L^2 x_t \\
x_{t+1} &= L^{-1} x_t \\
\Delta x_t &= (1 - L)\, x_t
\end{aligned}
$$

Lag operator

$$
\begin{aligned}
x_t &= \rho x_{t-1} + \varepsilon_t \text{ with } |\rho| < 1 \\
x_t &= \rho L x_t + \varepsilon_t \\
(1 - \rho L)\, x_t &= \varepsilon_t \\
x_t &= \frac{\varepsilon_t}{1 - \rho L}
\end{aligned}
$$

Lag operator

$$
\begin{aligned}
\frac{1}{1-\rho} &= 1+\rho+\rho^2+\rho^3+\cdots \text{ if } |\rho|<1 \\
\frac{1}{1-\rho L} &= 1+\rho L+\rho^2 L^2+\rho^3 L^3+\cdots \text{ if } |\rho L|<1
\end{aligned}
$$

Why go to frequency domain

1. Extract that part from the data that your model tries to explain
   - e.g., business cycle frequencies
2. Some calculations are easier in frequency domain
   - e.g., auto-covariances of ARMA processes
   - not the focus on this lecture

Fourier Transform

Given a sequence $\{x_j\}_{-\infty}^{\infty}$ the Fourier transform is defined as

$$F(\omega) = \sum_{j=-\infty}^{\infty} x_j e^{-i\omega j}$$

If $x_j = x_{-j}$ then

$$F(\omega) = x_0 + \sum_{j=1}^{\infty} x_j \left(e^{-i\omega j} + e^{i\omega j}\right) = x_0 + \sum_{j=1}^{\infty} 2x_j \cos(\omega j)$$

and the Fourier transform is a real-valued symmetric function.

Fourier Transform

Given a continuous sequence $x\left(j\right)$, the Fourier transform is defined as

$$F(\omega) = \int_{j=-\infty}^{\infty} x\left(j\right) e^{-i\omega j} dj$$

Fourier Transform

- It is just a definition!
  - which turns out to be useful
  - see links to youtube videos at the end of the slides

Why useful?

The Fourier transform can *detect* frequency of data. Suppose, the sequences considered are time series. Specifically, consider

$$\begin{aligned} x_{A,t} &= \cos(\omega_A t) \\ x_{B,t} &= \cos(\omega_B t) \\ x_t &= x_{A,t} + x_{B,t} \end{aligned}$$

Why useful?

$$
\begin{aligned}
& F_A(\omega) \\
= & \int_{-\infty}^{\infty} \cos(\omega_A t)\, e^{-i\omega t} dt \\
= & \int_{-\infty}^{\infty} \cos(\omega_A t)\left(\cos(-\omega t) + i \sin(-\omega t)\right) dt \\
= & \int_{-\infty}^{\infty} \cos(\omega_A t)\cos(-\omega t)\, dt + i \int_{-\infty}^{\infty} \cos(\omega_A t)\sin(-\omega t)\, dt \\
= & \int_{-\infty}^{\infty} \cos(\omega_A t)\cos(\omega t)\, dt - i \int_{-\infty}^{\infty} \cos(\omega_A t)\sin(-\omega t)\, dt \\
= & \begin{cases} > 0 \text{ if } \omega = \pm\omega_A \\ = 0 \text{ if } \omega \neq \omega_A \end{cases}
\end{aligned}
$$

Why useful?

Because things are additive, we get

$$\begin{aligned} F(\omega) &= F_A(\omega) + F_B(\omega) \\ &= \begin{cases} > 0 \text{ if } \omega = \pm\omega_A \\ > 0 \text{ if } \omega = \pm\omega_B \\ = 0 \text{ otherwise} \end{cases} \end{aligned}$$

- $\Longrightarrow$ Fourier transform of a time series selects the frequencies.
- If there is a phase shift in any of these series, then the series are no longer symmetric around $t = 0$ and the Fourier transform would also have an imaginary part.

Inverse Fourier Transform

- Given a Fourier Transform $F(\omega)$, one can back out the original sequence using

$$x_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)e^{i\omega j}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)\left(\cos\omega j + i\sin\omega j\right)d\omega$$

and if $F(\omega)$ is symmetric then

$$x_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)\cos\omega j\; d\omega = \frac{1}{\pi}\int_{0}^{\pi} F(\omega)\cos\omega j\; d\omega$$

- derivation is in the notes

Thinking differently about a time series

- You can take the Fourier transform of any sequence
- So you can also take it of a time series
  - take finite analogue if time series is finite

Thinking differently about a time series

- Finite Fourier transform of $\{x_t\}_{t=1}^{T}$, scaled by $\sqrt{T}$

$$\tilde{x}(\omega) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} e^{-i\omega t} x_t.$$

- Let

$$\omega_j = (j-1)2\pi/T \text{ for } j = 1, \cdots, T$$

Formulas

$\tilde{x}(\omega)$ can be expressed as

$$\tilde{x}(\omega) = |\tilde{x}(\omega)| \, e^{i\phi(\omega)}$$

with

$$\tilde{x}(-\omega) = |\tilde{x}(\omega)| \, e^{-i\phi(\omega)}$$

Thinking differently about a time series

- The *finite* inverse Fourier transform is given by

$$
\begin{aligned}
x_t &= \frac{1}{\sqrt{T}} \sum_{|\omega_j| \leq \pi} e^{i\omega_j t} \tilde{x}(\omega_j) \\
&= \frac{1}{\sqrt{T}} \sum_{|\omega_j| \leq \pi} |\tilde{x}(\omega)| \, e^{i\omega_j t} e^{i\phi(\omega)} \\
&= \frac{\tilde{x}(0) + \sum_{0<\omega_j \leq \pi} |\tilde{x}(\omega)| \left(e^{i(\omega_j t + \phi(\omega)}) + e^{-i(\omega_j t + \phi(\omega))} \right)}{\sqrt{T}}
\end{aligned}
$$

Thinking differently about a time series

Using

$$e^{i\delta(\omega)} = \cos\delta(\omega) + i\sin\delta(\omega)$$

or

$$e^{i\delta(\omega)} + e^{i\delta(\omega)} = 2\cos\delta(\omega)$$

gives

$$x_t = \frac{1}{\sqrt{T}}\left(\tilde{x}(0) + 2\sum_{0<\omega_j\leq\pi}\left|\tilde{x}(\omega_j)\right|\cos(\omega_j t + \phi(\omega_j))\right)$$

This makes clear we can think of a time series as a sum of deterministic cosine waves

- $\left|\tilde{x}(\omega_j)\right|$ captures the quantiative importance of a particular frequency

Two realizations of the same time series process

- Suppose x_t and y_t are two realizations of the same time series process, say an AR(1).
- Then the $\left|\tilde{x}(\omega_j)\right|$ would be similar (or same if there is no sampling uncertainty)
- The random differences due to different realizations of the shocks are captured by having different phase shifts, $\phi(\omega)$

Thinking differently about a time series

$$x_t = \frac{1}{\sqrt{T}} \left(\tilde{x}(0) + 2 \sum_{\omega_j \leq \pi} \left| \tilde{x}(\omega_j) \right| \cos(\omega_j t + \phi(\omega_j)) \right)$$

Simple interpretation:

- x_t : dependent variable (T observations)
- $\omega_j t$: T independent variables
- get perfect fit by choosing $\left|\tilde{x}(\omega_j)\right|$ and $\phi(\omega_j)$
- if $\left|\tilde{x}(\omega_j)\right|$ is high than that frequency is important for time variation x_t

(Informally) thinking about the variance

- What is the variance of x_t?
- Focus on the case with $\mathsf{E}[x_t] = 0$
- $\mathsf{E}\left[x_t^2\right]$ depends on $\mathsf{E}\left[\left(\sum_{\omega_j<\pi} \tilde{x}(\omega_j)\right)^2\right]$
- Fortunately, $\lim_{T\longrightarrow\infty} \mathsf{E}\left[\tilde{x}(\omega_j)\tilde{x}(\omega_i)\right] = 0$
- variance of x_t depends just on sum of the squared $\left|\tilde{x}(\omega_j)\right|$ terms (or on the integral in the limit)

Spectrum

- Given a sequence $\{\gamma_j\}_{-\infty}^{\infty}$ of autocovariances of a scalar process then the spectrum is defined as

$$S(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j} = \frac{1}{2\pi} \left(\gamma_0 + \sum_{j=1}^{\infty} 2\gamma_j \cos(\omega j) \right)$$

- And according to the inverse

$$\gamma_0 = \int_{-\pi}^{\pi} S(\omega)\, d\omega$$

- That is, if you integrate over all frequencies you get the variance. This is also consistent with the view that the data can be thought of as a sum of cosine waves

Spectrum

So spectrum is just the Fourier transform of the covariances

Spectrum of transformed series

If

$$y_t = \sum_{j=-\infty}^{\infty} b_j x_{t-j} = b(L)x_t$$

Then

$$S_y(\omega) = b(e^{-i\omega})b(e^{i\omega})S_x(\omega) = \left|b(e^{-i\omega})\right|^2 S_x(\omega)$$

- $|\cdot|$ is the modulus of the complex number
- Note that $b(e^{-i\omega})$ is the Fourier transform of the b_j sequence
- For symmetric filters we have $b(e^{-i\omega}) = b(e^{i\omega})$

Examples - white noise

$$
\begin{aligned}
x_t &= \varepsilon_t \text{ and } \mathrm{E}\left[\varepsilon_t \varepsilon_{t-j}\right] = 0 \text{ for } j \neq 0 \\
S(\omega) &= \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j} \\
&= \frac{1}{2\pi} \left(\gamma_0 + \sum_{j=1}^{\infty} 2\gamma_j \cos(\omega j) \right) \\
&= \frac{\gamma_0}{2\pi}
\end{aligned}
$$

Examples - AR(1)

$$
\begin{aligned}
y_t &= \rho y_{t-1} + x_t \\
y_t &= \frac{x_t}{1-\rho L} \\
S_y(\omega) &= \left| \frac{1}{1-\rho e^{-i\omega}} \right|^2 S_x(\omega) \\
&= \frac{1}{1-\rho e^{-i\omega}} \quad \frac{1}{1-\rho e^{+i\omega}} S_x(\omega) \\
&= \frac{1}{1-\rho\left(e^{-i\omega}+e^{+i\omega}\right)+\rho^2} S_x(\omega) \\
&= \frac{1}{1-2\rho\cos\omega+\rho^2} S_x(\omega)
\end{aligned}
$$

Examples - VAR(P)

$$
\begin{aligned}
y_t &= \sum_{j=1}^{J} A_j y_{t-j} + x_t \\
y_t &= \left(I - \sum_{j=1}^{J} A_j L^j \right)^{-1} x_t \\
S_y(\omega) &= \left(I - \sum_{j=1}^{J} A_j e^{-ij\omega} \right)^{-1} S_x(\omega) \left(I - \sum_{j=1}^{J} A_j' e^{ij\omega} \right)^{-1}
\end{aligned}
$$

Examples - VAR(P)

$$
\begin{aligned}
y_t &= \sum_{j=1}^{J} A_j y_{t-j} + x_t \\
S_y(0) &= \left(I - \sum_{j=1}^{J} A_j e^{-ij\times 0} \right)^{-1} S_x(\omega) \left(I - \sum_{j=1}^{J} A_j' e^{ij\times 0} \right)^{-1} \\
&= \left(I - \sum_{j=1}^{J} A_j \right)^{-1} S_x(\omega) \left(I - \sum_{j=1}^{J} A_j' \right)^{-1}
\end{aligned}
$$

This last formula is useful in calculating Heteroskedastic and

Autocorrelation Consistent (HAC) variance-covariance estimators

What is a filter?

- A filter is just a tranformation of the data
- Typically with a particular purpose
  - e.g. remove seasonality or "noise"
- Filters can be expressed as

$$\begin{aligned} x_t^f &= b(L)x_t \\ b(L) &= \sum_{j=-\infty}^{\infty} b_j L^j \end{aligned}$$

Examples of filters

$$
\begin{aligned}
b(L) &= 1 - L \Longrightarrow x_t^f = x_t - x_{t-1} \\
b(L) &= -\frac{1}{2}L^{-1} + 1 - \frac{1}{2}L
\end{aligned}
$$

I(0) and I(1) processes

- I(0) processes do not have trends[1]
- x_t is I(1) if Δx_t is I(0)

[1] I(0) processes are often referred to as stationary stationary processes. Strictly speaking that is not correct. Stationary processes also don't have trends but also don't allow for changes in the distribution. Defining I(0) is a bit tricky and there are different definitions. A relatively easy description of an I(0) process is a stochastic process for which past errors do not accumulate.

I(0) and I(1) processes

$$x_t = B(L)\,\varepsilon_t,$$

where ε_t is white noise.

- If x_t is I(1), then $B(1) = \infty$
- If $B(1) < \infty$, then x_t is I(0)

(same holds is ε_t is I(0))

Filters that induce stationarity, meaning I(0)

- Suppose that x_t is I(1). Thus

$$(1-L)x_t = z_t$$

  with z_t an I(0) process.
- Filtering gives

$$x_t^f = b(L)x_t$$

- Question: When is x_t^f I(0)?

Filters that induce stationarity

Define $\bar{b}(L)$ such that

$$b(L) = (1-L)\bar{b}(L).$$

If

$$\bar{b}(1) < \infty,$$

then $x_t^f = b(L)x_t$ is I(0) even if x_t is I(1).

$$\begin{aligned} x_t^f &= b(L)x_t \\ &= (1-L)\bar{b}(L)x_t \\ &= (1-L)\bar{b}(L)\frac{z_t}{(1-L)} \\ &= \bar{b}(L)z_t \end{aligned}$$

Filters that induce stationarity

Suppose

$$b(L) = \sum_{j=-J}^{J} b_j L^j \text{ and } b\left(1\right) = 0$$

That is, $L = 1$ is a root of the problem $b\left(L\right) = 0$, which means we have

$$b(L) = (1 - L)\bar{b}(L).$$

Since $\bar{b}(L)$ is of *finite*-order, we know that

$$\bar{b}(1) < \infty.$$

Consequently, $x_t^f = b(L)x_t$ is I(0) if x_t is I(1).

Spectrum of filtered series

$$y_t = \sum_{j=-\infty}^{\infty} b_j x_{t-j} = b(L)x_t$$

Then

$$S_y(\omega) = b(e^{-i\omega})b(e^{i\omega})S_x(\omega) = \left|b(e^{-i\omega})\right|^2 S_x(\omega)$$

- $|\cdot|$ is the modulus of the complex number
- Note that $b(e^{-i\omega})$ is the Fourier transform of the b_j sequence
- For symmetric filters we have $b(e^{-i\omega}) = b(e^{i\omega})$

Band-pass filters

$$y_t = b(L)x_t$$

Goal:

$$S_y(\omega) = \begin{cases} S_x(\omega) & \text{if } \omega_L \leq |\omega| \leq \omega_H \\ 0 & \text{o.w.} \end{cases}$$

Thus we need

$$b(e^{-i\omega}) = \begin{cases} 1 & \text{if } \omega_L \leq |\omega| \leq \omega_H \\ 0 & \text{o.w.} \end{cases}$$

- How to find the coefficients b_j that correspond with this?
- Since $b(e^{-i\omega})$ is a Fourier transform we can use the inverse of the Fourier transform

Coefficients of band-pass filters

Inverse of the Fourier transform for b_0:

$$
\begin{aligned}
b_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} b(e^{-i\omega})e^{i\omega j}d\omega \\
&= \frac{1}{2\pi}\left(\int_{-\omega_H}^{-\omega_L} 1\times e^{i\omega j}d\omega + \int_{\omega_L}^{\omega_H} 1\times e^{i\omega j}d\omega\right) \\
&= \frac{\omega_H - \omega_L}{\pi}
\end{aligned}
$$

Coefficients of band-pass filters

Inverse of the Fourier transform for b_j:

$$
\begin{aligned}
b_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} b(e^{-i\omega})e^{i\omega j}d\omega \\
&= \frac{1}{2\pi}\left(\int_{-\omega_H}^{-\omega_L} 1\times e^{i\omega j}d\omega + \int_{\omega_L}^{\omega_H} 1\times e^{i\omega j}d\omega\right) \\
&= \frac{1}{2\pi}\left(\int_{\omega_L}^{\omega_H}\left(e^{i\omega j}+e^{-i\omega j}\right)d\omega\right) \\
&= \frac{1}{2\pi}\int_{\omega_L}^{\omega_H} 2\cos(\omega j)d\omega \\
&= \frac{1}{\pi}\frac{1}{j}\sin\omega j]_{\omega_L}^{\omega_H} = \frac{\sin(\omega_H j)-\sin(\omega_L j)}{\pi j}
\end{aligned}
$$

Note that you can also get b_0 from the last line by using l'Hopital's rule

Properties of the band-pass filter

$$b(L) = \sum_{j=-\infty}^{\infty} b_j L^j$$

- Consider the roots to the problem:

$$b(L) = 0$$

If $L = 1$ is a root of the problem, then we have

$$b(L) = (1 - L)\bar{b}(L) \quad \text{with } \bar{b}(1) < \infty$$

Properties of the band-pass filter

- But $L = 1$ is a root of our filter as long as $\omega_L > 0$, because then we have by construction

$$b(1) = b(e^{-i0}) = 0$$

Clearly, if you do not filter out the zero frequency then you do not induce stationarity

More on I(1) processes

- Discussion above showed

$$x_t^f = b(L)x_t \text{ is I(0) even if } x_t \text{ is I(1)}$$

- This is not enough to show that the filter does what it is supposed to do, which is
  - ensure the spectrum of the filtered series is zero for the excluded frequencies
  - ensure the spectrum of the filtered series equals the spectrum of the original series for the included frequencies
- The second condition requires a definition of the spectrum for I(1) processes

Spectrum for I(1) processes

Consider an arbitrary I(1) process

$$x_t = \frac{z_t}{1-L}$$

Let

$$x_{\rho,t} = \frac{z_t}{1-\rho L}$$

For $\rho < 1$ the spectrum of $x_{\rho,t}$ is well defined

$$S_{\rho,x}(\omega) = \frac{1}{1-2\rho\cos(\omega)+\rho^2} S_z(\omega)$$

Define the spectrum of x_t as

$$S_x(\omega) = \lim_{\rho \longrightarrow 1} S_{\rho,x}(\omega)$$

This is well defined for all $\omega > 0$, but not for $\omega = 0$.

Filtered I(1) process

$$x_t^f = b(L)x_t$$

Let $b(L)$ be a band-pass filter, that is,

$$b(e^{-i\omega}) = \begin{cases} 1 & \text{if } \omega_L \leq \omega \leq \omega_H \\ 0 & \text{o.w.} \end{cases}$$

Filtered I(1) process

- if $\omega_L > 0$, then it can be shown that
  - x_t^f is I(0) (because as shown above we know that $b(1) = 0$) and
  - $S_{x^f}(\omega) = \begin{cases} S_x(\omega) & \text{if } \omega_L \leq \omega \leq \omega_H \\ 0 & \text{o.w.} \end{cases}$
- That is, using the definition of the Spectrum for I(1) processes the filter does exactly what it is supposed to do
- Proof is simple; The only tricky thing is to prove is that

$$b(e^{-i0})S_x(0) = 0$$

Practical Filter

- The filter constructed so far is two-sided and infinite order
- Implementable version would be to use

$$\tilde{b}(L) = \sum_{j=-J}^{J} b_j L^j$$

But it is not necessarily the case that

$$\tilde{b}(1) = 0$$

So instead use

$$a(L) = \sum_{j=-J}^{J} a_j L^j$$

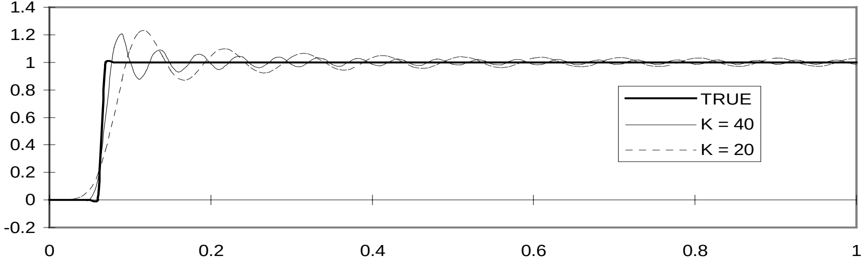
with

$$a_j = b_j + \mu \quad \text{and} \quad \mu = -\frac{\sum_{j=-J}^{J} b_j}{2J+1}$$

Properties practical filter

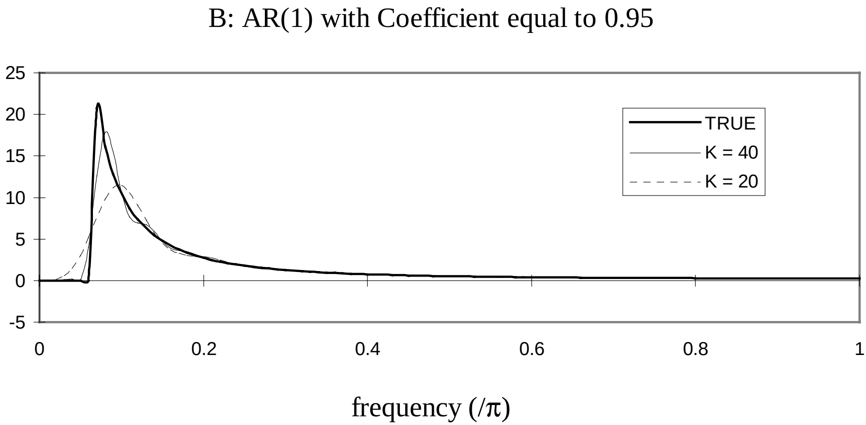
FIGURE 3.1: SPECTRA OF FILTERED PROCESSES ($\omega_1 = \pi/16, \omega_2 = \pi$)

A: White Noise (Squared Gain)

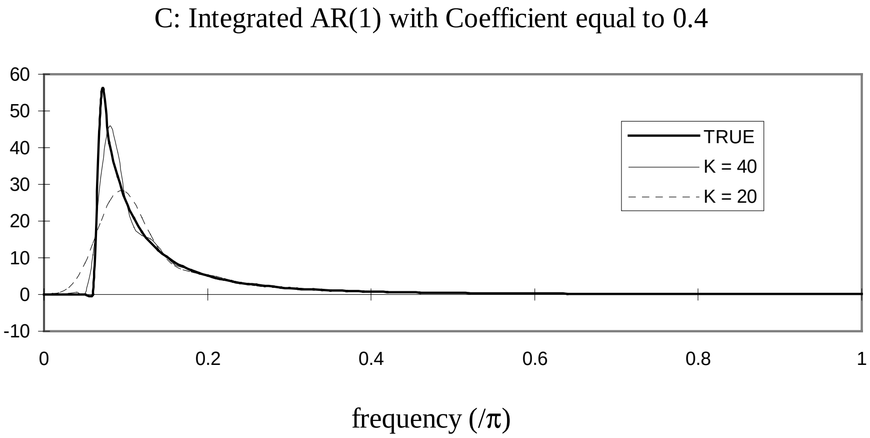


frequency (/π)

Properties practical filter



Properties practical filter



Hodrick-Prescott Filter

- The HP trend, $x_{\tau,t}$ is defined as follows

$$\{x_{\tau,t}\}_{t=1}^{T} = \arg\min_{\{x_{\tau,t}\}_{t=1}^{T}} \sum_{t=2}^{T-1} (x_t - x_{\tau,t})^2 + \lambda \sum_{t=2}^{T-1} \left\{ \begin{bmatrix} (x_{\tau,t+1} - x_{\tau,t}) \\ -(x_{\tau,t} - x_{\tau,t-1}) \end{bmatrix}^2 \right\}$$

Hodrick-Prescott Filter

- $\lambda = 1,600$ standard for quarterly data
- the HP filter is then approximately equal to a band-pass filter with $\omega_L = \pi/16$ and $\omega_H = \pi$.
    - That is, it keeps that part of the series associated with cycles that have a period less than 32 ($=2\pi/(\pi/16)$) periods (i.e. quarters).

Understanding filtered data is tricky

- Is filtered white noise serially uncorrelated?
- Are the filtered price level and filtered output positively correlated in a model with only demand shocks? (example below is from Den Haan 2000)

Simple demand shock model

- Output is demand determined

$$y_t = y_t^d = D_t - P_t$$

- Demand is given by

$$\begin{aligned}(1-\lambda_1 L)(1-\lambda_2 L)(1-\lambda_3 L) D_t &= \varepsilon_t \\ \text{with } -1 &< \lambda_3 < \lambda_2 < \lambda_1 \leq 1\end{aligned}$$

- Output is given by

$$y_t^s = a + bt$$

Simple demand shock model

- Equilibrium price level $\widetilde{P}_t$ satisfies

$$\widetilde{P}_t = D_t - y_t^s$$

- Actual prices adjust gradually

$$P_t = (1-\beta)\, P_{t-1} + \beta\widetilde{P}_t$$

Simple demand shock model

Solution

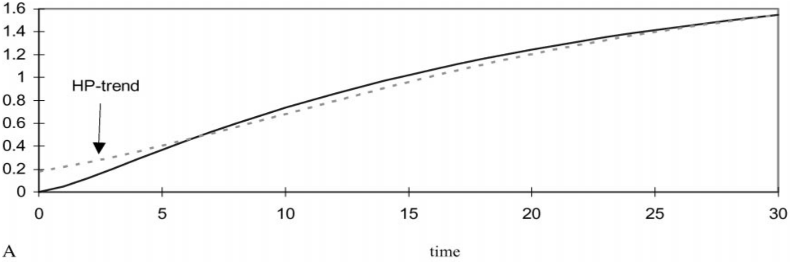
- Price level

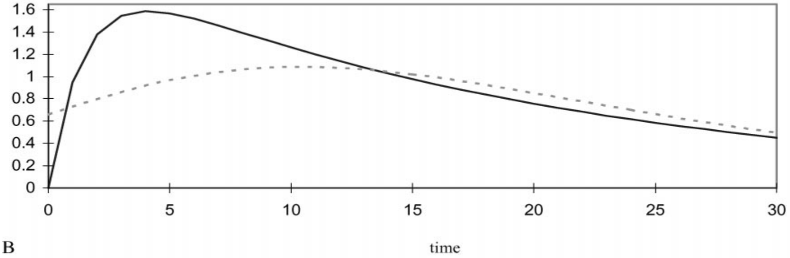
$$P_t = \frac{\beta \varepsilon_t}{(1-(1-\beta)L)(1-\lambda_1 L)(1-\lambda_2 L)(1-\lambda_3 L)}$$

- Output

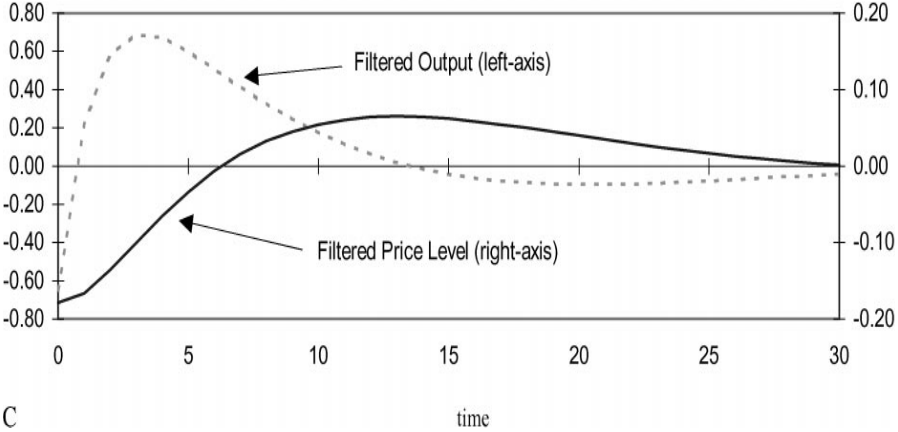
$$y_t = \frac{(1-\beta)(1-L)\varepsilon_t}{(1-(1-\beta)L)(1-\lambda_1 L)(1-\lambda_2 L)(1-\lambda_3 L)}$$

Positive correlation for unfiltered series





Negative correlation for filtered series



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Cool youtube videos

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- https://www.youtube.com/watch?v=1JnayXHhjlg
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